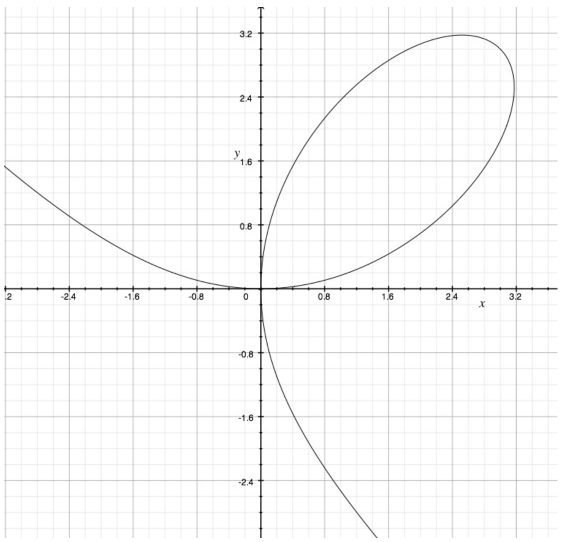
Carleton UNIVERSITY
Canada's Capital University

Mathematics Tutorial Series
Differential Calculus #14

Implicit Functions and Derivatives



This curve is called the **Folium of Descartes**

$$x^3 + y^3 = 6xy$$

This equation is not of the form "y = something".

It is called a "relation".

It tells us how x and y are related but doesn't let us directly calculate y for a given x.

We still want to know things like max and min points, slopes of tangents, intervals of increase or decrease. So it would be helpful to have a way to get the derivative

$$\frac{dy}{dx}$$

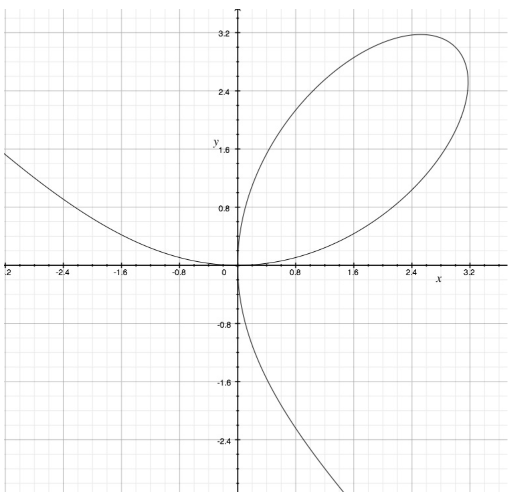
It is simple. Just differentiate both sides. Use all the standard rules but stop whenever you hit y' since this is what we are looking for.

$$x^3 + y^3 = 6xy$$

$$3x^2 + 3y^2y' = 6y + 6xy'$$

Solve this for y' to get:

$$(3y^2 - 6x)y' = 6y - 3x^2$$
$$y' = \frac{6y - 3x^2}{3y^2 - 6x}$$



For example $y' = 0$ when

$$6y = 3x^2$$

and

$$x^3 + y^3 = 6xy$$

Chase the algebra and you get two solutions.
Either $(x, y) = (0,0)$ or $(x, y) = (2\sqrt[3]{2}, 2\sqrt[3]{4})$.

More examples

[1] $x^2 + y^2 = r^2$

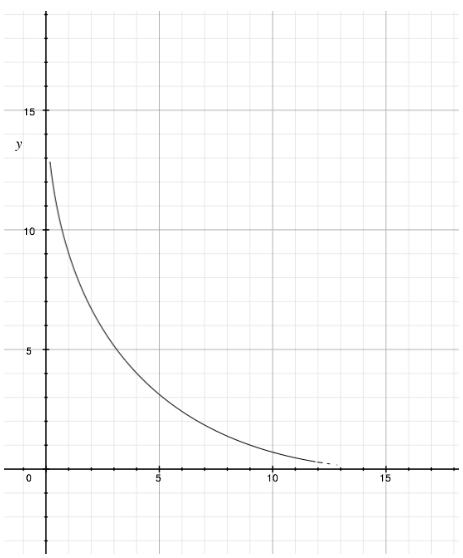
This is a circle with radius r.
Implicit differentiation gives the rate of change as:

$$2x + 2y\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$ to get:

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

[2] $\sqrt{x} + \sqrt{y} = 4$



Find the slop of the tangent

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0$$

$$y' = -\frac{\sqrt{y}}{\sqrt{x}}$$

[3] $\boldsymbol{y = x^x}$

We use a method called **logarithmic differentiation**

From $\quad y = x^x$
Go to $\quad \log y = \log x^x = x \log x$

Differentiate implicitly to get

$$\frac{1}{y} y' = \log x + x \left(\frac{1}{x}\right)$$

So

$$y' = (1 + \log x) x^x$$

Note:

Writing $1 + \log x$ is not so error prone as $\log x + 1$

[4] $\boldsymbol{y = x^{\cos x}}$

Use logarithmic differentiation because the exponent includes the variable.

From: $y = x^{\cos x}$
Go to: $\log y = \log(x^{\cos x}) = \cos x \log x$

Differentiate implicitly using the product rule

$$(\log y)' = (\cos x \log x\ )'$$

$$\frac{1}{y} y' = -\sin x \log x + \cos x \left(\frac{1}{x}\right)$$

$$y' = \left(-\sin x\ \log x + \frac{\cos x}{x}\right) x^{\cos x}$$

Summary

1. Implicit differentiation starts with an equation.
2. You take derivatives on both sides of the "=".
3. The target, y', will arise from chain rule calculations.
4. Solve for y'.
5. If the variable is used in the exponent take logs of both sides to clear the exponents. Then just do implicit differentiation.

Example

Ideal Gas Law

The ideal gas law is a relation between

- Pressure P
- Volume V
- Number of molecules of gas m
- Absolute temperature T
- With a constant R

$$PV = mRT$$

Suppose one or more of these quantities is changing with time t.

So some rate of change is given and we want to know how the other variables will change.

This is called a "related rates" problem.

Use implicit differentiation.
Calculate derivatives with respect to t on both sides:

$$\frac{dPV}{dt} = \frac{dmRT}{dt}$$

$$P'V + PV' = R(m'T + mT')$$

If we know that the volume is constant and the amount of gas is fixed then $V' = 0$ and $m' = 0$.

So we get: $P'V = mRT'$.

This tells us how increasing temperature is related to increasing pressure.